MATH4030 Differential Geometry, 2017-18

Solutions to Midterm

Q1. Let $\alpha : (-1,1) \to \mathbb{R}^3$ be the space curve given by

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right).$$

(a) Show that the curve $\alpha(s)$ is parametrized by arc length.

Solution: Differentiating $\alpha(s)$ w.r.t. s, we have

$$\alpha'(s) = \left(\frac{1}{2}(1+s)^{1/2}, -\frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Therefore, the length is

$$\|\alpha'(s)\| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}} = 1.$$

Hence, α is parametrized by arc length.

(b) Compute the curvature $k(s)$ and torsion $\tau(s)$ of the curve $\alpha(s)$.

Solution: Since α is p.b.a.l., we have the unit tangent vector

$$T(s) = \alpha'(s) = \left(\frac{1}{2}(1+s)^{1/2}, -\frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Differentiating once w.r.t. s, we obtain

$$T'(s) = \left(\frac{1}{4}(1+s)^{-1/2}, \frac{1}{4}(1-s)^{-1/2}, 0\right).$$

Therefore, the curvature is

$$k(s) = \|T'(s)\| = \sqrt{\frac{1}{16}(1+s)^{-1} + \frac{1}{16}(1-s)^{-1}} = \frac{1}{2\sqrt{2}\sqrt{1-s^2}}.$$

Note that $k(s) > 0$ for all $s \in (-1,1)$. The unit normal is

$$N(s) = \frac{1}{k(s)}T'(s) = \left(\frac{1}{\sqrt{2}}(1-s)^{1/2}, \frac{1}{\sqrt{2}}(1+s)^{1/2}, 0\right).$$

The binormal is then given by

$$B(s) = T(s) \times N(s) = \left(-\frac{1}{2}(1+s)^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Differentiating w.r.t. s,

$$B'(s) = \left(-\frac{1}{4}(1+s)^{-1/2}, -\frac{1}{4}(1-s)^{-1/2}, 0\right).$$

Therefore, the torsion is given by

$$\tau(s) = \langle B'(s), N(s)\rangle = -\frac{1}{4\sqrt{2}}\left(\sqrt{\frac{1-s}{1+s}} + \sqrt{\frac{1+s}{1-s}}\right) = -\frac{1}{2\sqrt{2}\sqrt{1-s^2}}.$$

Q2. Let $\alpha = \alpha(s) : \mathbb{R} \to \mathbb{R}^2$ be a plane curve p.b.a.l. such that all the tangent lines of α pass through a fixed point $\mathbf{p}_0 \in \mathbb{R}^2$. Show that α must be a straight line passing through the point $\mathbf{p}_0$.

Solution: By assumption, there exists a smooth function $f(s) : \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(s) + f(s)\alpha'(s) \equiv \mathbf{p}_0 \quad \text{for all } s \in \mathbb{R}.$$

Differentiating the above identity w.r.t. s,

$$\alpha'(s) + f'(s)\alpha'(s) + f(s)\alpha''(s) \equiv \mathbf{0}.$$

As α is p.b.a.l., $T(s) = \alpha'(s)$ and Frenet's equation gives $\alpha''(s) = T'(s) = k(s)N(s)$. Therefore, we have

$$(1 + f'(s))T(s) + f(s)k(s)N(s) \equiv \mathbf{0}.$$

Since $\{T(s), N(s)\}$ is an orthonormal basis at each s, we have

$$1 + f'(s) \equiv 0 \quad \text{and} \quad f(s)k(s) \equiv 0.$$

The first equations implies that $f(s) = -s + C$ for some constant $C \in \mathbb{R}$. Plug it into the second equation and by continuity of $k(s)$ we must have $k(s) \equiv 0$. By the fundamental theorem of plane curve, $\alpha(s)$ must be a straight line. Since the tangent lines of a straight line agree the line itself, $\alpha(s)$ must also pass through the point $\mathbf{p}_0$.

Q3. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve described in polar coordinates by $r = r(\theta)$, i.e.

$$\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad \theta \in I.$$

(a) Show that for any $[a, b] \subset I$, $L_a^b(\alpha) = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$.

Solution: Differentiating $\alpha(\theta)$ w.r.t. θ, we have

$$\alpha'(\theta) = \Big(r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta\Big).$$

Therefore, the length squared is given by

$$\begin{aligned}\|\alpha'(\theta)\|^2 &= (r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2 \\ &= r'(\theta)^2 + r(\theta)^2.\end{aligned}$$

By definition of arc length, we have

$$L_a^b(\alpha) = \int_a^b \|\alpha'(\theta)\|\, d\theta == \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta.$$

(b) Show that the curvature at $\theta \in I$ is given by

$$k(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{3/2}}.$$

Hint: Recall that the curvature of a plane curve $\beta(t)$ not necessarily parametrized by arc length is given by the formula

$$k(t) = \frac{\det(\beta'(t), \beta''(t))}{|\beta'(t)|^3}.$$

Solution: See Problem Set 2.

(c) Suppose there exists $\theta_0 \in I$ such that $r(\theta_0) \geq r(\theta)$ for all $\theta \in I$. Prove that $k(\theta_0) \geq 0$.

Solution: Since θ_0 is a maximal of the function $r(\theta)$, we have

$$r'(\theta_0) = 0 \quad \text{and} \quad r''(\theta_0) \leq 0.$$

Combining this with the result in (b), and that $r(\theta) \geq 0$ by definition of polar coordinates, we have

$$k(\theta_0) = \frac{r(\theta_0)^2 - r(\theta_0)r''(\theta_0)}{r(\theta)^3} \geq 0.$$

Q4. Let $S \subset \mathbb{R}^3$ be a surface and $p_0 \in S$. Suppose $a \in \mathbb{S}^2$ is a unit vector perpendicular to $T_{p_0}S$ and let P be the plane through p_0 perpendicular to a. Define the function $f : S \to \mathbb{R}^3$ by

$$f(p) = p - \langle p - p_0, a\rangle a.$$

(a) Show that $f(S) \subset P$ and $f : S \to P$ is a smooth map between surfaces.

Solution: To check that $f(S) \subset P$, pick any $p \in S$, we want to show that $\langle f(p) - p_0, a\rangle = 0$, i.e. $f(p) \in P = \{q \in \mathbb{R}^3 : \langle q - p_0, a\rangle = 0\}$. Since a is a unit vector,

$$\langle f(p) - p_0, a\rangle = \langle p - p_0, a\rangle - \langle p - p_0, a\rangle = 0.$$

This proves $f(S) \subset P$. To see that $f : S \to P$ is a smooth map between surfaces, we just observe that f is a well-defined smooth function for all $p \in \mathbb{R}^3$, hence its restriction to S is a smooth map.

(b) Compute the differential $df_{p_0} : T_{p_0}S \to T_{p_0}P \cong T_{p_0}S$.

Solution: Note that $T_{p_0}S = \{v \in \mathbb{R}^3 : \langle v, a\rangle = 0\} = T_{p_0}P$. Take any $v \in T_{p_0}S$, there exists a smooth curve $\alpha(s) : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p_0$ and $\alpha'(0) = v$. By the definition of differential, we have

$$\begin{aligned} df_{p_0}(v) &= \left.\frac{d}{dt}\right|_{t=0} \Big(\alpha(t) - \langle \alpha(t) - p_0, a\rangle a\Big) \\ &= \alpha'(0) - \langle \alpha'(0), a\rangle a \\ &= v - \langle v, a\rangle a \\ &= v. \end{aligned}$$

The last equality follows from the fact that $v \perp a$ by the definition of $T_{p_0}S$ and a. Therefore, $df_{p_0} : T_{p_0}S \to T_{p_0}P \cong T_{p_0}S$ is the identity map on $T_{p_0}S$.

(c) Prove that S is locally a graph over the plane P near p_0.

Solution: Notice that f is the orthogonal projection onto the plane P. As df_{p_0} is the identity map on $T_{p_0}S$, which is clearly a linear isomorphism, by Inverse Function Theorem , $f : S \to P$ is a local diffeomorphism near p_0. The inverse of such a local diffeomorphism that gives S as a graph over P locally near p_0.

Q5. Let $S \subset \mathbb{R}^3$ be the half-cone given by

$$S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, \ z > 0\}.$$

(a) Show that S is a regular surface.

Solution:
Let $F : \mathbb{R}^3 \setminus \{z \leq 0\} \to \mathbb{R}$ be the smooth function defined by $F(x, y, z) = x^2 + y^2 - z^2$. Then we have

$$\nabla F(x, y, z) = (2x, 2y, -2z).$$

Note that $\nabla F = \mathbf{0}$ only when $(x, y, z) = \mathbf{0}$, which does not lie on the surfaces S (as $z > 0$). Therefore, we have $S = F^{-1}(0)$ is the level set of a regular value of F, hence must be a regular surface.

(b) Show that $X(u, v) = (u \cos v, u \sin v, u)$ where $(u, v) \in (0, \infty) \times (0, 2\pi)$ is a parametrization of S.

Solution: Note first that $(u \cos v)^2 + (u \sin v)^2 = u^2$, hence $X(u, v) \in S$ for all $(u, v) \in (0, \infty) \times (0, 2\pi)$. It is clear that X is smooth. Moreover,

$$X_u = (\cos v, \sin v, 1) \quad \text{and} \quad X_v = (-u \sin v, u \cos v, 0),$$

which are linearly independent everywhere (for example, by comparing the z-component). It remains to check that X is bijective onto its image, the rest the follows from the inverse function theorem (for surfaces). To see X is one-to-one, suppose $X(u, v) = X(u', v')$. Then, the z-component gives $u = u'$. The first two components together with the restriction $v, v' \in (0, 2\pi)$ then implies that $v = v'$ as well. Therefore, X is injective. This proves the assertion that X is a parametrization of S.

(c) Compute the mean curvature H and Gauss curvature K of S (with respect to the unit normal N that points "into" the cone).

Solution: Taking second derivatives of X, we obtain

$$X_{uu} = (0, 0, 0), \quad X_{uv} = X_{vu} = (-\sin v, \cos v, 0) \quad \text{and} \quad X_{vv} = (-u \cos v, -u \sin v, 0).$$

The unit normal is obtained by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{1}{\sqrt{2}}(-\cos v, -\sin v, 1),$$

which points into the cone. Therefore, the first fundamental form and its inverse are given by

$$(g_{ij}) = \begin{pmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & u^2 \end{pmatrix},$$

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{u^2} \end{pmatrix}.$$

On the other hand, the second fundamental formula is

$$(A_{ij}) = \begin{pmatrix} \langle X_{uu}, N\rangle & \langle X_{uv}, N\rangle \\ \langle X_{vu}, N\rangle & \langle X_{vv}, N\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{u}{\sqrt{2}} \end{pmatrix}.$$

Applying the local formula for H and K, we have

$$H = \operatorname{tr}((g^{ij})(A_{ij})) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}u} \end{pmatrix} = \frac{1}{\sqrt{2}u},$$

$$K = \frac{\det(A_{ij})}{\det(g_{ij})} = 0.$$